

G-RADICAL SUPPLEMENTED MODULES

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Abstract

In this work, g-radical supplemented modules which is a proper generalization of g-supplemented modules are defined and some properties of these modules are investigated. It is proved that the finite sum of g-radical supplemented modules is g-radical supplemented. It is also proved that every factor module and every homomorphic image of a g-radical supplemented module is g-radical supplemented. Let R be a ring. Then ${}_R R$ is g-radical supplemented if and only if every finitely generated R -module is g-radical supplemented. In the end of this work, it is given two examples for g-radical supplemented modules separating with g-supplemented modules.

Key words: Small Submodules, Radical, Supplemented Modules, Radical (Generalized) Supplemented Modules.

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1 INTRODUCTION

In this paper, all rings are associative with identity and all modules are unital left modules.

Let M be an R -module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a small submodule of M , denoted by $N \ll M$. Let M be an R -module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, then N is called a direct summand of M and it is denoted by $M = N \oplus K$. For any module M , we have $M = M \oplus 0$. The intersection of all maximal submodules of an R -module M is called the radical of M and denoted by $RadM$. If M have no maximal submodules, then we call $RadM = M$. A submodule N of an R -module M is called an essential submodule and denoted by $N \leq M$ in case $K \cap N \neq 0$ for

every submodule $K \neq 0$. Let M be an R -module and K be a submodule of M . K is called a generalized small submodule of M denoted by $K \ll_g M$ if for every essential submodule T of M with the property $M = K + T$ implies that $T = M$. It is clear that every small submodule is a generalized small submodule but the converse is not true generally. Let M be an R -module. M is called an hollow module if every proper submodule of M is small in M . M is called local module if M has a largest submodule, i.e. a proper submodule which contains all other proper submodules. Let U and V be submodules of M . If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a supplement of U in M . M is called a supplemented module if every submodule of M has a supplement in M . Let M be an R -module and $U, V \leq M$. If $M = U + V$ and $M = U + T$ with $T \leq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V \ll_g M$, then V is called a g-supplement of U in M . M is called g-supplemented if every submodule of M has a g-supplement in M . The intersection of maximal essential submodules of an R -module M is called a generalized radical of M and denoted by $\text{Rad}_g M$. If M have no maximal essential submodules, then we denote $\text{Rad}_g M = M$.

Lemma 1.1 *Let M be an R -module and $K, L, N, T \leq M$. Then the followings are hold. [3,5]*

- (1) *If $K \leq N$ and N is generalized small submodule of M , then K is a generalized small submodule of M .*
- (2) *If K is contained in N and a generalized small submodule of N , then K is a generalized small submodule in submodules of M which contains submodule N .*
- (3) *Let $f : M \rightarrow N$ be an R -module homomorphism. If $K \ll_g M$, then $f(K) \ll_g M$.*
- (4) *If $K \ll_g L$ and $N \ll_g T$, then $K + N \ll_g L + T$.*

Corollary 1.2 *Let $M_1, M_2, \dots, M_n \leq M$, $K_1 \ll_g M_1$, $K_2 \ll_g M_2$, ..., $K_n \ll_g M_n$. Then $K_1 + K_2 + \dots + K_n \ll_g M_1 + M_2 + \dots + M_n$.*

Corollary 1.3 *Let M be an R -module and $K \leq N \leq M$. If $N \ll_g M$, then $N/K \ll_g M/K$.*

Corollary 1.4 *Let M be an R -module, $K \ll_g M$ and $L \leq M$. Then $(K + L)/L \ll_g M/L$.*

Lemma 1.5 *Let M be an R -module. Then $\text{Rad}_g M = \sum_{L \ll_g M} L$.*

Proof. See[3]. ■

Lemma 1.6 *The following assertions are hold.*

- (1) *If M is an R -module, then $Rm \ll_g M$ for every $m \in \text{Rad}_g M$.*
- (2) *If $N \leq M$, then $\text{Rad}_g N \leq \text{Rad}_g M$.*
- (3) *If $K, L \leq M$, then $\text{Rad}_g K + \text{Rad}_g L \leq \text{Rad}_g (K + L)$.*

(4) If $f : M \longrightarrow N$ is an R -module homomorphism, then $f(Rad_g M) \leq Rad_g N$.

(5) If $K, L \leq M$, then $Rad_g \frac{K+L}{L} \leq \frac{Rad_g K+L}{L}$.

Proof. Clear from Lemma 1.1 and Lemma 1.5. ■

Lemma 1.7 Let $M = \oplus_{i \in I} M_i$. Then $Rad_g M = \oplus_{i \in I} Rad_g M_i$.

Proof. Since $M_i \leq M$, then by Lemma 1.6(2), $Rad_g M_i \leq Rad_g M$ and $\oplus_{i \in I} Rad_g M_i \leq Rad_g M$. Let $x \in Rad_g M$. Then by Lemma 1.6(1), $Rx \ll_g M$. Since $x \in M = \oplus_{i \in I} M_i$, there exist $i_1, i_2, \dots, i_k \in I$ and $x_{i_1} \in M_{i_1}$, $x_{i_2} \in M_{i_2}$, \dots , $x_{i_k} \in M_{i_k}$ such that $x = x_{i_1} + x_{i_2} + \dots + x_{i_k}$. Since $Rx \ll_g M$, then by Lemma 1.1(4), under the canonical epimorphism π_{i_t} ($t = 1, 2, \dots, k$) $Rx_{i_t} = \pi_{i_t}(Rx) \ll_g Rx_{i_t}$. Then $x_{i_t} \in Rad_g M_{i_t}$ ($t = 1, 2, \dots, k$) and $x = x_{i_1} + x_{i_2} + \dots + x_{i_k} \in \oplus_{i \in I} Rad_g M_i$. Hence $Rad_g M \leq \oplus_{i \in I} Rad_g M_i$ and since $\oplus_{i \in I} Rad_g M_i \leq Rad_g M$, $Rad_g M = \oplus_{i \in I} Rad_g M_i$. ■

2 G-RADICAL SUPPLEMENTED MODULES

Definition 2.1 Let M be an R -module and $U, V \leq M$. If $M = U + V$ and $U \cap V \leq Rad_g V$, then V is called a generalized radical supplement (briefly, g -radical supplement) of U in M . If every submodule of M has a generalized radical supplement in M , then M is called a generalized radical supplemented (briefly, g -radical supplemented) module.

Clearly we see that every g -supplemented module is g -radical supplemented. But the converse is not true in general (See Example 2.20 and Example 2.21).

Lemma 2.2 Let M be an R -module and $U, V \leq M$. Then V is a g -radical supplement of U in M if and only if $M = U + V$ and $Rm \ll_g V$ for every $m \in U \cap V$.

Proof. (\Rightarrow) Since V is a g -radical supplement of U in M , $M = U + V$ and $U \cap V \leq Rad_g V$. Let $m \in U \cap V$. Since $U \cap V \leq Rad_g V$, $m \in Rad_g V$. Hence by Lemma 1.6(1), $Rm \ll_g V$.

(\Leftarrow) Since $Rm \ll_g V$ for every $m \in U \cap V$, then by Lemma 1.6(1), $U \cap V \leq Rad_g V$ and hence V is a g -radical supplement of U in M . ■

Lemma 2.3 Let M be an R -module, $M_1, U, X \leq M$ and $Y \leq M_1$. If X is a g -radical supplement of $M_1 + U$ in M and Y is a g -radical supplement of $(U + X) \cap M_1$ in M_1 , then $X + Y$ is a g -radical supplement of U in M .

Proof. Since X is a g -radical supplement of $M_1 + U$ in M , $M = M_1 + U + X$ and $(M_1 + U) \cap X \leq Rad_g X$. Since Y is a g -radical supplement of $(U + X) \cap M_1$ in M_1 , $M_1 = (U + X) \cap M_1 + Y$ and $(U + X) \cap Y = (U + X) \cap M_1 \cap Y \leq Rad_g Y$. Then $M = M_1 + U + X = M_1 = (U + X) \cap M_1 + Y + U + X = U + X + Y$ and, by Lemma 1.6(3), $U \cap (X + Y) \leq (U + X) \cap Y + (U + Y) \cap X \leq Rad_g Y + (M_1 + U) \cap X \leq Rad_g Y + Rad_g X \leq Rad_g (X + Y)$. Hence $X + Y$ is a g -radical supplement of U in M . ■

Lemma 2.4 *Let $M = M_1 + M_2$. If M_1 and M_2 are g -radical supplemented, then M is also g -radical supplemented.*

Proof. Let $U \leq M$. Then 0 is a g -radical supplement of $M_1 + M_2 + U$ in M . Since M_1 is g -radical supplemented, there exists a g -radical supplement X of $(M_2 + U) \cap M_1 = (M_2 + U + 0) \cap M_1$ in M_1 . Then by Lemma 2.3, $X + 0 = X$ is a g -radical supplement of $M_2 + U$ in M . Since M_2 is g -radical supplemented, there exists a g -radical supplement Y of $(U + X) \cap M_2$ in M_2 . Then by Lemma 2.3, $X + Y$ is a g -radical supplement of U in M . ■

Corollary 2.5 *Let $M = M_1 + M_2 + \dots + M_k$. If M_i is g -radical supplemented for every $i = 1, 2, \dots, k$, then M is also g -radical supplemented.*

Proof. Clear from Lemma 2.4. ■

Lemma 2.6 *Let M be an R -module, $U, V \leq M$ and $K \leq U$. If V is a g -radical supplement of U in M , then $(V + K)/K$ is a g -radical supplement of U/K in M/K .*

Proof. Since V is a g -radical supplement of U in M , $M = U + V$ and $U \cap V \leq \text{Rad}_g V$. Then $M/K = U/K + (V + K)/K$ and by Lemma 1.6(5), $(U/K) \cap ((V + K)/K) = (U \cap V + K)/K \leq (\text{Rad}_g V + K)/K \leq \text{Rad}_g (V + K)/K$. Hence $(V + K)/K$ is a g -radical supplement of U/K in M/K . ■

Lemma 2.7 *Every factor module of a g -radical supplemented module is g -radical supplemented.*

Proof. Clear from Lemma 2.6. ■

Corollary 2.8 *The homomorphic image of a g -radical supplemented module is g -radical supplemented.*

Proof. Clear from Lemma 2.7. ■

Lemma 2.9 *Let M be a g -radical supplemented module. Then every finitely M -generated module is g -radical supplemented.*

Proof. Clear from Corollary 2.5 and Corollary 2.8. ■

Corollary 2.10 *Let R be a ring. Then ${}_R R$ is g -radical supplemented if and only if every finitely generated R -module is g -radical supplemented.*

Proof. Clear from Lemma 2.9. ■

Theorem 2.11 *Let M be an R -module. If M is g -radical supplemented, then $M/\text{Rad}_g M$ is semisimple.*

Proof. Let $U/\text{Rad}_g M \leq M/\text{Rad}_g M$. Since M is g-radical supplemented, there exists a g-radical supplement V of U in M . Then $M = U + V$ and $U \cap V \leq \text{Rad}_g V$. Thus $M/\text{Rad}_g M = U/\text{Rad}_g M + (V + \text{Rad}_g M)/\text{Rad}_g M$ and $(U/\text{Rad}_g M) \cap ((V + \text{Rad}_g M)/\text{Rad}_g M) = (U \cap V + \text{Rad}_g M)/\text{Rad}_g M \leq (\text{Rad}_g V + \text{Rad}_g M)/\text{Rad}_g M = \text{Rad}_g M/\text{Rad}_g M = 0$. Hence $M/\text{Rad}_g M = U/\text{Rad}_g M \oplus (V + \text{Rad}_g M)/\text{Rad}_g M$ and $U/\text{Rad}_g M$ is a direct summand of M . ■

Lemma 2.12 *Let M be a g-radical supplemented module and $L \leq M$ with $L \cap \text{Rad}_g M = 0$. Then L is semisimple. In particular, a g-radical supplemented module M with $\text{Rad}_g M = 0$ is semisimple.*

Proof. Let $X \leq L$. Since M is g-radical supplemented, there exists a g-radical supplement T of X in M . Hence $M = X + T$ and $X \cap T \leq \text{Rad}_g T \leq \text{Rad}_g M$. Since $M = X + T$ and $X \leq L$, by Modular Law, $L = L \cap M = L \cap (X + T) = X + L \cap T$. Since $X \cap T \leq \text{Rad}_g M$ and $L \cap \text{Rad}_g M = 0$, $X \cap L \cap T = L \cap X \cap T \leq L \cap \text{Rad}_g M = 0$. Hence $L = X \oplus L \cap T$ and X is a direct summand of L . ■

Proposition 2.13 *Let M be a g-radical supplemented module. Then $M = K \oplus L$ for some semisimple module K and some module L with essential generalized radical.*

Proof. Let K be a complement of $\text{Rad}_g M$ in M . Then by [8, 17.6], $K \oplus \text{Rad}_g M \trianglelefteq M$. Since $K \cap \text{Rad}_g M = 0$, then by Lemma 2.12, K is semisimple. Since M is g-radical supplemented, there exists a g-radical supplement L of K in M . Hence $M = K + L$ and $K \cap L \leq \text{Rad}_g L \leq \text{Rad}_g M$. Then by $K \cap \text{Rad}_g M = 0$, $K \cap L = 0$. Hence $M = K \oplus L$. Since $M = K \oplus L$, then by Lemma 1.7, $\text{Rad}_g M = \text{Rad}_g K \oplus \text{Rad}_g L$. Hence $K \oplus \text{Rad}_g M = K \oplus \text{Rad}_g L$. Since $K \oplus \text{Rad}_g L = K \oplus \text{Rad}_g M \trianglelefteq M = K \oplus L$, then by [1, Proposition 5.20], $\text{Rad}_g L \trianglelefteq L$. ■

Proposition 2.14 *Let M be an R -module and $U \leq M$. the following statements are equivalent.*

- (1) *There is a decomposition $M = X \oplus Y$ with $X \leq U$ and $U \cap Y \leq \text{Rad}_g Y$.*
- (2) *There exists an idempotent $e \in \text{End}(M)$ with $e(M) \leq U$ and $(1 - e)(U) \leq \text{Rad}_g(1 - e)(M)$.*
- (3) *There exists a direct summand X of M with $X \leq U$ and $U/X \leq \text{Rad}_g(M/X)$.*
- (4) *U has a g-radical supplement Y such that $U \cap Y$ is a direct summand of U .*

Proof. (1) \Rightarrow (2) For a decomposition $M = X \oplus Y$, there exists an idempotent $e \in \text{End}(M)$ with $X = e(M)$ and $Y = (1 - e)(M)$. Since $e(M) = X \leq U$, we easily see that $(1 - e)(U) = U \cap (1 - e)(M)$. Then by $Y = (1 - e)(M)$ and $U \cap Y \leq \text{Rad}_g Y$, $(1 - e)(U) = U \cap (1 - e)(M) = U \cap Y \leq \text{Rad}_g Y = \text{Rad}_g(1 - e)(M)$.

(2) \Rightarrow (3) Let $X = e(M)$ and $Y = (1 - e)(M)$. Since $e \in \text{End}(M)$ is idempotent, we easily see that $M = X \oplus Y$. Then $M = U + Y$. Since $e(M) =$

$X \leq U$, we easily see that $(1-e)(U) = U \cap (1-e)(M)$. Since $M = U + Y$ and $U \cap Y = U \cap (1-e)(M) = (1-e)(U) \leq \text{Rad}_g(1-e)(M) = \text{Rad}_g Y$, Y is a g-radical supplement of U in M . Then by Lemma 2.6, $M/X = (Y+X)/X$ is a g-radical supplement of U/X in M/X . Hence $U/X = (U/X) \cap (M/X) \leq \text{Rad}_g(M/X)$.

(3) \Rightarrow (4) Let $M = X \oplus Y$. Since $X \leq U$, $M = U + Y$. Let $t \in U \cap Y$ and $Rt + T = Y$ for an essential submodule T of Y . Let $((T+X)/X) \cap (L/X) = 0$ for a submodule L/X of M/X . Then $(L \cap T + X)/X = ((T+X)/X) \cap (L/X) = 0$ and $L \cap T + X = X$. Hence $L \cap T \leq X$ and since $X \cap Y = 0$, $L \cap T \cap Y \leq X \cap Y = 0$. Since $L \cap Y \cap T = L \cap T \cap Y = 0$ and $T \leq Y$, $L \cap Y = 0$. Since $X \leq L$ and $M = X + Y$, by Modular Law, $L = L \cap M = L \cap (X + Y) = X + L \cap Y = X + 0 = X$. Hence $L/X = 0$ and $(T+X)/X \leq M/X$. Since $Rt + T = Y$, $R(t+X) + (T+X)/X = (Rt+X)/X + (T+X)/X = (Rt+T+X)/X = (Y+X)/X = M/X$. Since $t \in U$, $t+X \in U/X \leq \text{Rad}_g(M/X)$ and hence $R(t+X) \ll_g M/X$. Then by $R(t+X) + (T+X)/X$ and $(T+X)/X \leq M/X$, $(T+X)/X = M/X$ and then $X + T = M$. Since $X + T = M$ and $T \leq Y$, by Modular Law, $Y = Y \cap M = Y \cap (X + T) = X \cap Y + T = 0 + T = T$. Hence $Rt \ll_g Y$ and by Lemma 2.2, Y is a g-radical supplement of U in M . Since $M = X \oplus Y$ and $X \leq U$, by Modular Law, $U = U \cap M = U \cap (X \oplus Y) = X \oplus U \cap Y$. Hence $U \cap Y$ is a direct summand of U .

(4) \Rightarrow (1) Let $U = X \oplus U \cap Y$ for a submodule X of U . Since Y is a g-radical supplement of U in M , $M = U + Y$ and $U \cap Y \ll_g Y$. Hence $M = U + Y = (X \oplus U \cap Y) + Y = X \oplus Y$. ■

Theorem 2.15 *Let V be a g-radical supplement of U in M . If U is a generalized maximal submodule of M , then $U \cap V$ is a unique generalized maximal submodule of V .*

Proof. Since U is a generalized maximal submodule of M and $V/(U \cap V) \simeq (V+U)/U = M/U$, $U \cap V$ is a generalized maximal submodule of V . Hence $\text{Rad}_g V \leq U \cap V$ and since $U \cap V \leq \text{Rad}_g V$, $\text{Rad}_g V = U \cap V$. Thus $U \cap V$ is a unique generalized maximal submodule of V . ■

Definition 2.16 *Let M be an R -module. If every proper essential submodule of M is generalized small in M or M has no proper essential submodules, then M is called a generalized hollow module.*

Clearly we see that every hollow module is generalized hollow.

Definition 2.17 *Let M be an R -module. If M has a large proper essential submodule which contain all essential submodules of M or M has no proper essential submodules, then M is called a generalized local module.*

Clearly we see that every local module is generalized local.

Proposition 2.18 *Generalized hollow and generalized local modules are g-supplemented, so are g-radical supplemented.*

Proof. Clear from definitions. ■

Proposition 2.19 *Let M be an R -module and $\text{Rad}_g M \neq M$. Then M is generalized hollow if and only if M is generalized local.*

Proof. (\implies) Let M be generalized hollow and let L be a proper essential submodule of M . Then $L \ll_g M$ and by Lemma 1.5, $L \leq \text{Rad}_g M$. Thus $\text{Rad}_g M$ is a proper essential submodule of M which contain all proper essential submodules of M .

(\impliedby) Let M be a generalized local module, T be a large essential submodule of M and L be a proper essential submodule of M . Let $L + S = M$ with $S \leq M$. If $S \neq M$, then $L + S \leq T \neq M$. Thus $S = M$ and $L \ll_g M$. ■

Example 2.20 *Consider the \mathbb{Z} -module \mathbb{Q} . Since $\text{Rad}_g \mathbb{Q} = \text{Rad} \mathbb{Q} = \mathbb{Q}$, $_{\mathbb{Z}}\mathbb{Q}$ is g -radical supplemented. But, since $_{\mathbb{Z}}\mathbb{Q}$ is not supplemented and every nonzero submodule of $_{\mathbb{Z}}\mathbb{Q}$ is essential in $_{\mathbb{Z}}\mathbb{Q}$, $_{\mathbb{Z}}\mathbb{Q}$ is not g -supplemented.*

Example 2.21 *Consider the \mathbb{Z} -module $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ for a prime p . It is easy to check that $\text{Rad}_g \mathbb{Z}_{p^2} \neq \mathbb{Z}_{p^2}$. By Lemma 1.7, $\text{Rad}_g (\mathbb{Q} \oplus \mathbb{Z}_{p^2}) = \text{Rad}_g \mathbb{Q} \oplus \text{Rad}_g \mathbb{Z}_{p^2} \neq \mathbb{Q} \oplus \mathbb{Z}_{p^2}$. Since \mathbb{Q} and \mathbb{Z}_{p^2} are g -radical supplemented, by Lemma 2.4, $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ is g -radical supplemented. But $\mathbb{Q} \oplus \mathbb{Z}_{p^2}$ is not g -supplemented.*

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